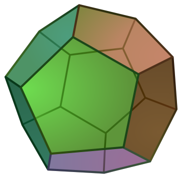
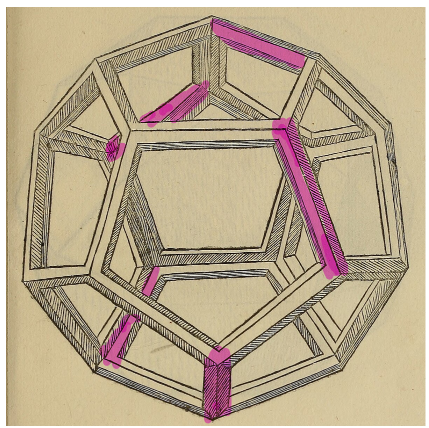
Higher-dimensional hyperbolic manifolds via Coxeter polytopes

Ventotene 2025



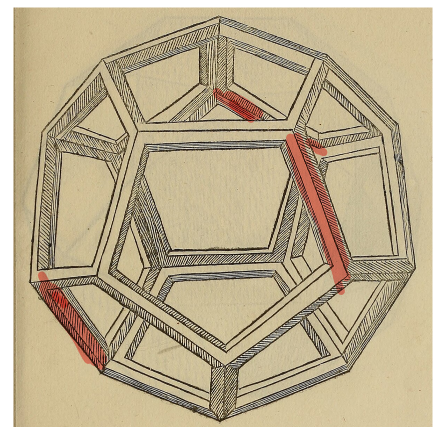
$\pi/3$	$2\pi/5$	$\pi/2$	$2\pi/3$
$\mathbb{H}^3$ ideal	$\mathbb{H}^3$	$\mathbb{H}^3$	$\mathbb{S}^3$

Seifert
Weber
1933



LARGE TURN

cycles of 5 $\rightsquigarrow \frac{2\pi}{5}$ $\mathbb{H}^3$

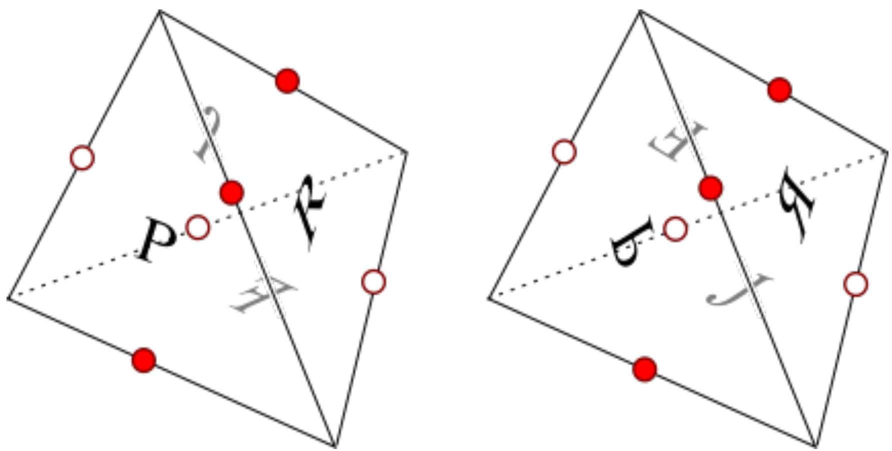


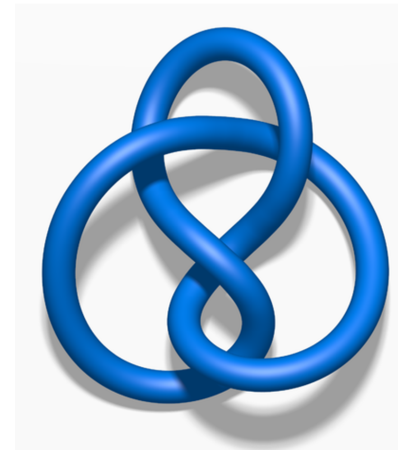
SHORT TURN

cycles of 3 $\rightsquigarrow \frac{2\pi}{3}$ $\mathbb{S}^3$

Poincaré
1903

Figure eight knot complement





It double covers a non-orientable 3-manifold

Gieseking 1912

Löbell 1931 Closed hyperbolic 3-manifold constructed using 8 right-angled dodecahedra

A generalization of his construction goes as follows:

$P \subseteq \mathbb{X}^n$ right-angled polyhedron $\qquad V \cong \left(\mathbb{Z}/2\mathbb{Z}\right)^N$.

vector space over $\mathbb{Z}/2\mathbb{Z}$

A COLOURING for P is a map $\lambda : \{\text{facets of } P\} \to V$ such that the following ADMISSIBILITY CONDITION holds:

$F_1 \cap \dots \cap F_m \neq \emptyset \implies \lambda(F_1), \dots, \lambda(F_m)$ independent

$\forall$ distinct facets $F_1, \dots, F_m$ of P

Recall: Let $F_1, \dots, F_k$ facets of P. $R_i :=$ reflection along F_i

whenever $F_i \cap F_j \neq \emptyset$

$$\Gamma = \langle R_1, \dots, R_k \mid R_i^2, \underset{[R_i, R_j]}{\overset{}{(R_i R_j)^2}} \rangle$$ discrete with fundamental domain P

A colouring λ gives a morphism $\Gamma \xrightarrow{\lambda} V$

$$R_i \longmapsto \lambda(F_i)$$

Prop: $\operatorname{Ker}\lambda$ is torsion-free

pf: Every torsion element $g \in \Gamma$ is conjugate to an isometry that fixes pointwise a face F of P

(suppose F maximal) $F = F_1 \cap \dots \cap F_m \Rightarrow g = R_1 \cdots R_m \Rightarrow \lambda(g) = \lambda(F_1) + \dots + \lambda(F_m) \neq 0$ $\square$

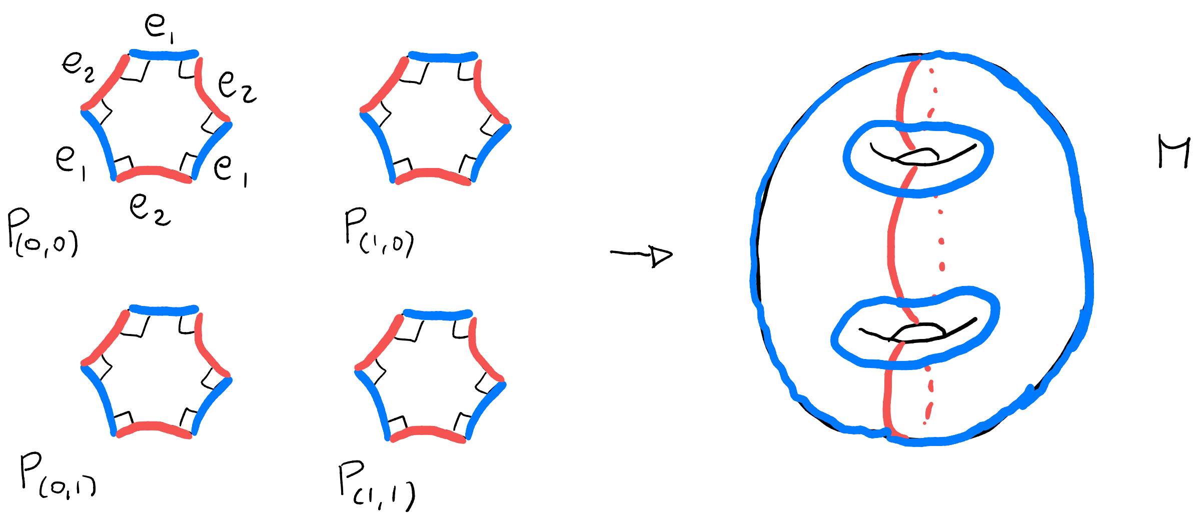
Defn: $\Gamma' = \operatorname{Ker}\lambda$ $M := \mathbb{X}^n / \Gamma'$ is a manifold!

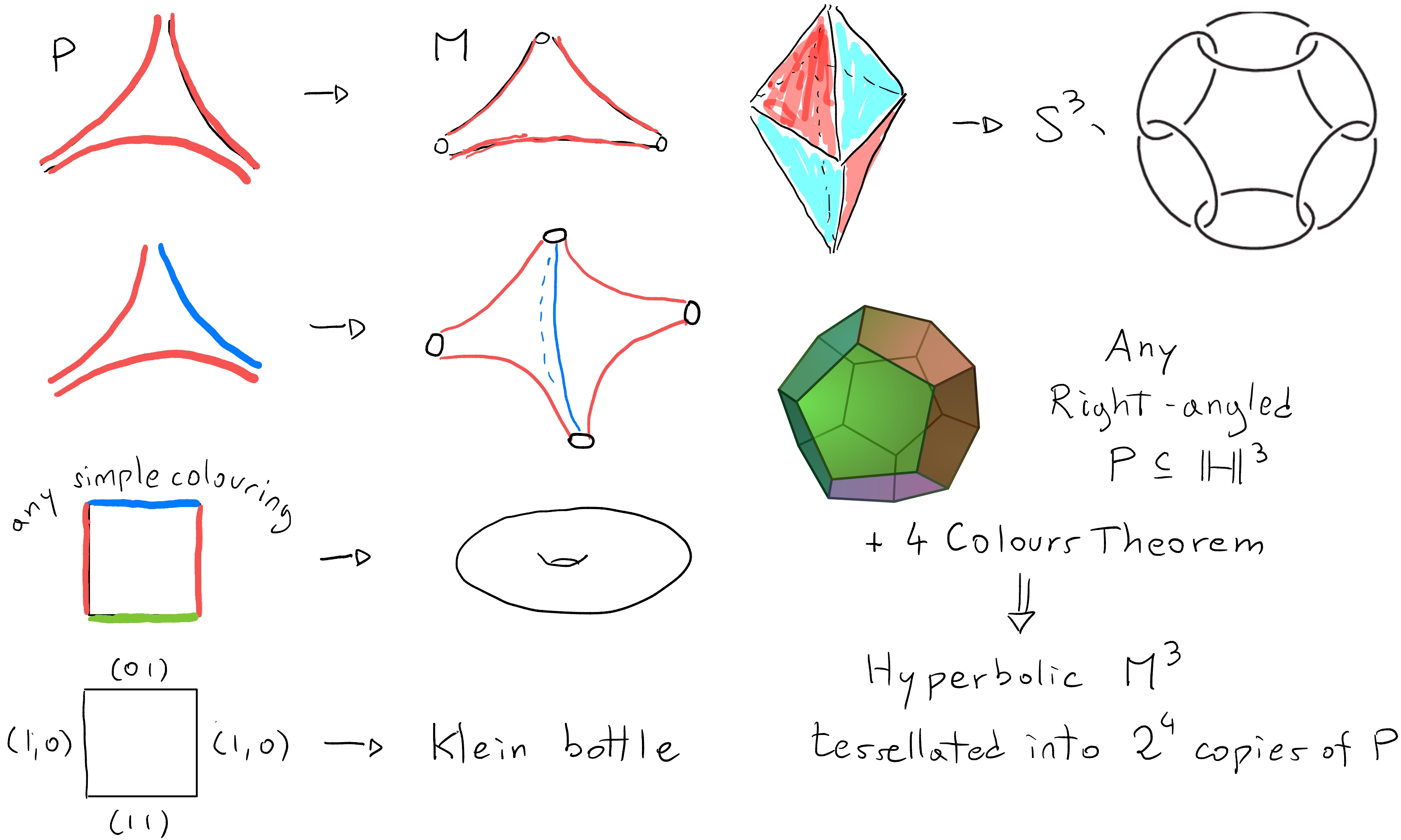
Can suppose $\lambda : \Gamma \to V$ is surjective. $N = \dim V$ $[\Gamma : \Gamma'] = 2^N$

M is tessellated into 2^N copies of P $M = \{P_v\}_{v \in V}$

facet F of P_v is identified with facet F of $P_{v+\lambda(F)}$

A colouring is SIMPLE if $V = (\mathbb{Z}/2\mathbb{Z})^N$, $\lambda(F_i) \in \{e_1, \dots, e_N\}$ $\forall i$

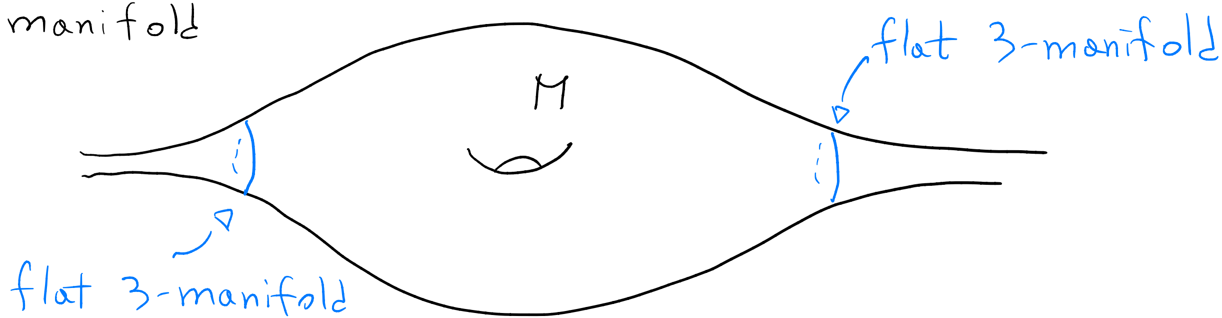




P
M
S^3,
any simple colouring
(01)
(1,0)
(1,0)
(11)
Klein bottle
Any Right-angled $P \subseteq \mathbb{H}^3$
+ 4 Colours Theorem
$\Downarrow$
Hyperbolic M^3
tessellated into 2^4 copies of P

Hyperbolic 4-Manifolds

M hyperbolic 4-manifold



$$\boxed{\mathrm{Vol}(M) = \frac{4}{3}\pi^2 \chi(M)}$$

Also true for orbifolds, Coxeter polyhedra

$$\chi(P) = \sum_{F \subseteq M} (-1)^{\dim F} \frac{1}{|\mathrm{stab}(F)|} = \sum_{F \subseteq M} \frac{(-1)^{\dim F}}{2^{n-\dim F}}$$

if P is right-angled

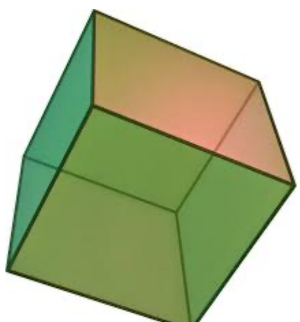
[Long, Reid] M^4 cusped hyperbolic with k cusps with sections $N_1, \dots, N_k$. Then

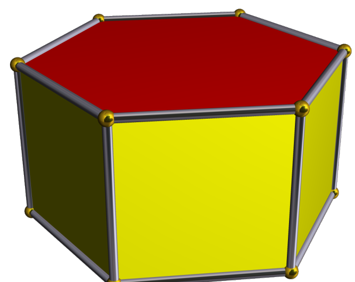
$$\boxed{\sigma(M) = \eta(N_1) + \dots + \eta(N_k)}$$

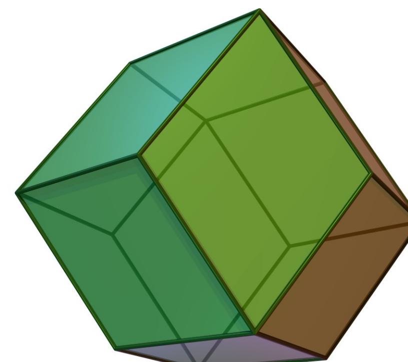
There are <u>6</u> <u>flat oriented 3-manifolds</u>:

Torus bundles over S^1 with monodromies

	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$	+ Hantsche-Wendt is a rational homology sphere
order:	1	2	4	3	6	
η:	0	0	1	$4/3$	$2/3$	0







<u>IDEAL RIGHT-ANGLED 24-CELL</u> $P \subseteq \mathbb{H}^4$, $\chi(P) = 1$

24 ideal <u>vertices</u> $\{(\pm 1, \pm 1, 0, 0)$ & all permutations$\}$

24 octahedral <u>facets</u>, orthogonal to vectors:

$$T_{24}^* = \underbrace{\{\pm 1, \pm i, \pm j, \pm k\}}_{Q_8} \cup \left\{\pm \frac{1}{2} \pm \frac{i}{2} \pm \frac{j}{2} \pm \frac{k}{2}\right\}$$

The 3 cosets of $Q_8 < T_{24}^*$ form a 3-colouring

$\rightsquigarrow$ M^4 with 24 cusps, all with 3-tori sections, $\chi(M) = 8$

a <u>very</u> symmetric 4-manifold

[Kolpakov, M.] There is a hyperbolic 4-manifold with 1 cusp

pf: cut and paste M along totally geodesic hypersurfaces

[Ferrari-Kolpakov-Slavich] $\exists$ M^4 with five HW's as cusp sections
(no rational homology at infinity)

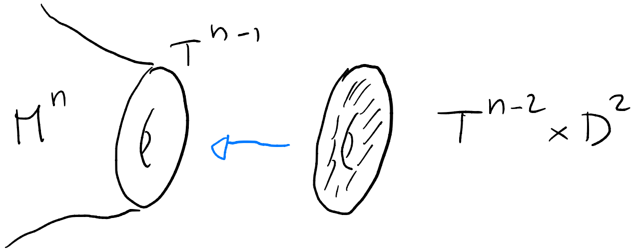
[Chen, Rizzi] $\exists$ M^4 cusp-transitive for all flat 3-manifolds

[Kolpakov, Riolo, Tschantz] $\exists$ M^4 cusped with $\sigma(M) \neq 0$

Question: Is there a hyp mfd with one cusp in dimension $n \geq 5$?

[Stover]: No arithmetic ones if n is large

Dehn filling



maintain the same colouring

$$\bar{M} = M \cup (T^{n-1} \times D^2)$$

$M \rightsquigarrow \bar{M}$ is a Dehn filling

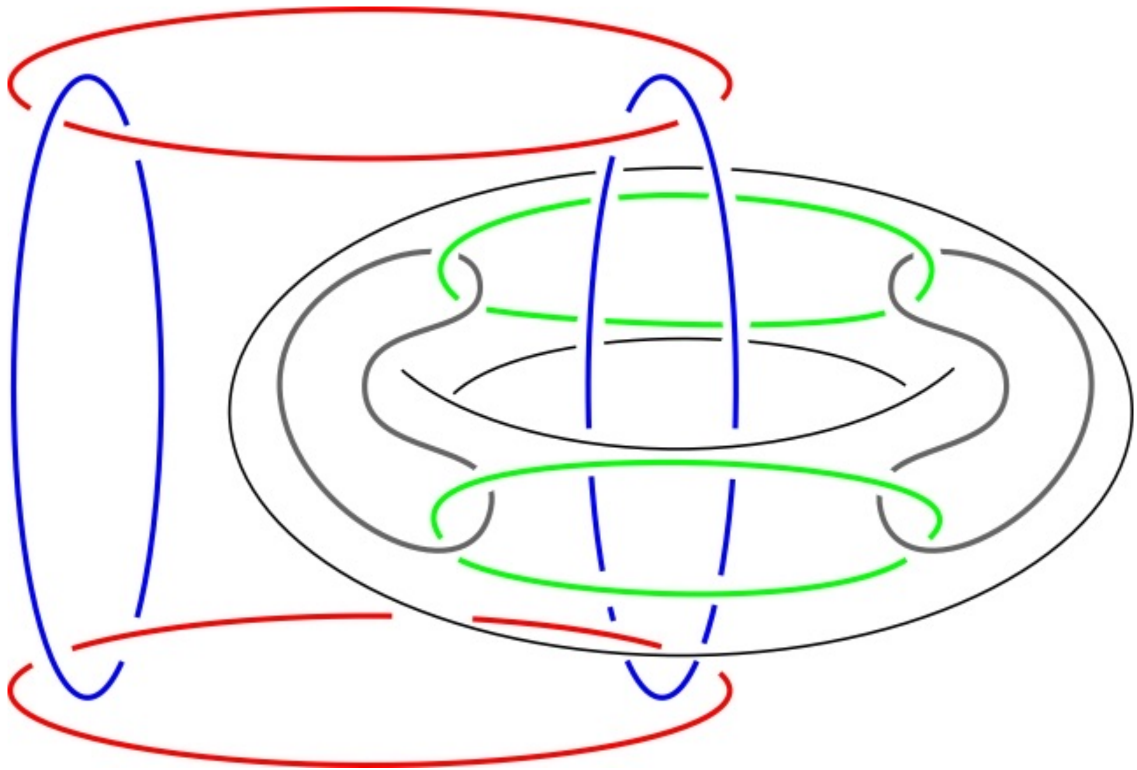
$$\chi(\bar{M}) = \chi(M)$$

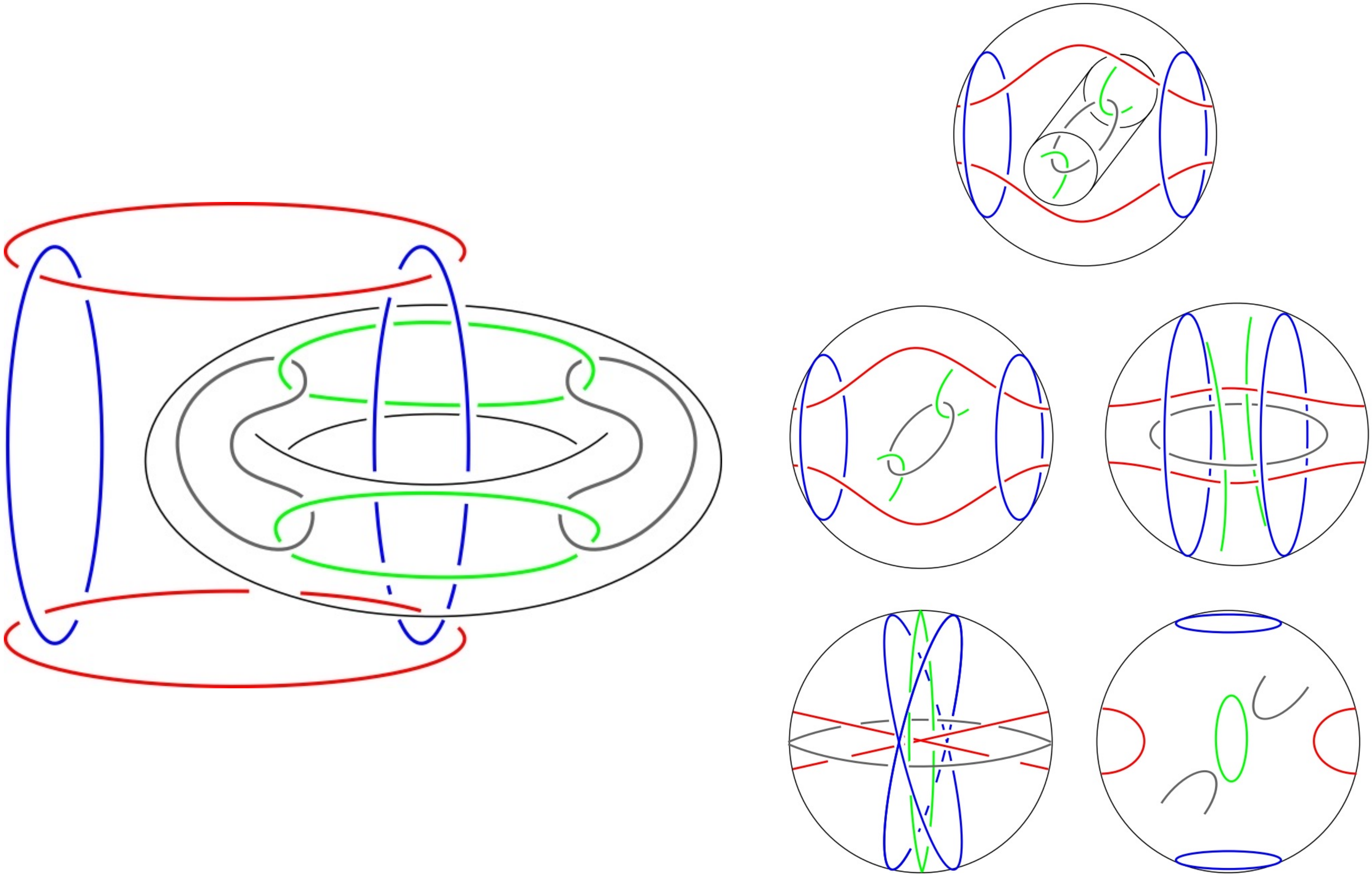
$$M = \bar{M} \setminus \{T_1 \sqcup \cdots \sqcup T_k\}$$ a link of $(n-2)$-tori in $\bar{M}$

Given $\bar{M}$, we can get only finitely many hyperbolic M by drilling some link of tori

[Ivansic] S^4 contains a hyperbolic link of 5 tori

A hyperbolic link of 5 tori in S^4





Right-angled hyperbolic polyhedra $P^3, P^4, P^5, P^6, P^7, P^8$

have minimal simple colourings with 3, 5, 8, 9, 14, 15 colours

we get manifolds $M^3, M^4, M^5, M^6, M^7, M^8$

	Euler	b_1	b_2	b_3	b_4	b_5	b_6	b_7	Cusps
M^3	0	3	2	0	0	0	0	0	3
M^4	2	5	10	4	0	0	0	0	5
M^5	0	24	120	136	39	0	0	0	40
M^6	−64	18	183	411	207	26	0	0	27
M^7	0	182	6321	41300	55139	24010	4031	0	4032
M^8	278528	365	33670	583290	1783226	1346030	456595	65279	65280

use formula of [Choi-Park] [Italiano, M., Migliorini]

M^3 = Borromean ring complement in S^3

M^4 = complement of five tori in S^4

Obs: If λ is a coloring for P, then $\dim V \geq n$.

If $\dim V = n$, then M is a SMALL COVERING for P

[Ratcliffe-Tschantz] Classification of SMALL COVERINGS of P^3, P^4, P^5, P^6

$\Gamma_0 = O^+(n,1) \cap SL(n+1, \mathbb{Z})$ fundamental domain = Coxeter simplex

$\Gamma_1 = \ker\left(\Gamma_0 \to SL(n+1, \mathbb{Z}/2\mathbb{Z})\right)$ " " " $= P^3, P^4, P^5, P^6$

Classification of torsion-free subgroups of Γ_1 of smallest index

By quotienting via symmetries, they found

N^4, N^5, N^6 smallest volume known

$\chi = 1$ -1

certainly smallest

Compact hyberbolic 4-manifolds

	$\frac{\pi}{3}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
120-cell		$\mathbb{H}^4$	$\mathbb{H}^4$	$\mathbb{H}^4$
		$\chi = 26$	$17/2$	1

Facets correspond to I^*_{120}

[Davis] Identify opposite facets of the 120-cell

Get ridges cycles of order 5 $\Rightarrow$ use $\frac{2\pi}{5}$ version

The DAVIS MANIFOLD has $\chi = 26$

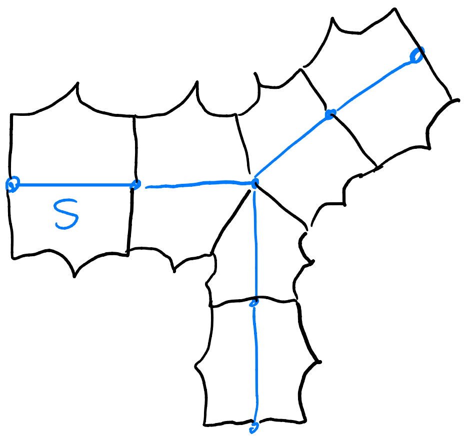
[Conder-Maclachlan] Examples with $\chi = 8$

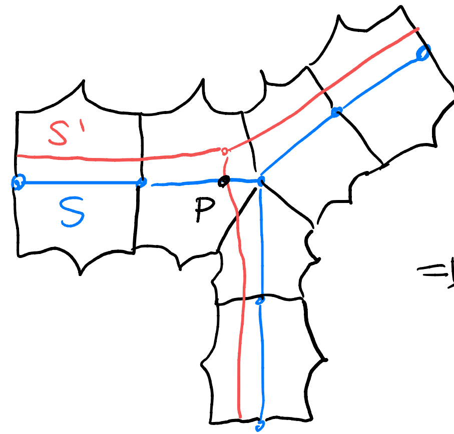
Use the $\pi/2$ version:

$I^*_{120} > T^*_{24}$ cosets give a 5-colouring $\Rightarrow$ another very symmetric 4-manifold

[M., Riolo, Slavich] There are NON SPIN compact hyp manifolds

pf:



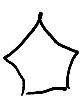


$\Rightarrow S \cap S' = \{P\}$

$\Downarrow$

$S \cdot S = 1$

 = right-angled pentagon

 = right-angled 120-cell

Colouring $\Rightarrow$ build M that contains that.

Odd Intersection form $\Rightarrow$ NOT SPIN